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# Some results on commutative semigroups and semigroup rings

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Let $G$ be a torsion-free abelian (additive) group, and let $S$ be a subsemigroup of $G$ which contains 0. Then $S$ is called a grading monoid ([No]). We will call a grading monoid simply a g-monoid.

For example, the direct sum $\mathbf{Z}_0 \oplus \cdots \oplus \mathbf{Z}_0$ of $n$-copies of the non-negative integers $\mathbf{Z}_0$ is a g-monoid.

Many terms in commutative ring theory may be defined analogously for $S$.

For example, a non-empty subset $I$ of $S$ is called an ideal of $S$ if $S+I \subset I$.

Let $I$ be an ideal of $S$ with $I \subsetneqq S$. If $s_1+s_2 \in I$ (for $s_1, s_2 \in S$) implies $s_1 \in I$ or $s_2 \in I$, then $I$ is called a prime ideal of $S$.

Let $\Gamma$ be a totally ordered abelian (additive) group. A mapping $v$ of a torsion-free abelian group $G$ onto $\Gamma$ is called a valuation on $G$ if $v(x+y) = v(x)+v(y)$ for all $x, y \in G$. The subsemigroup $\{x \in G \mid v(x) \geq 0\}$ of $G$ is called the valuation semigroup of $G$ associated to $v$.

The maximum number $n$ so that there exists a chain $P_1 \subsetneqq P_2 \subsetneqq \cdots \subsetneqq P_n$ of prime ideals of $S$ is called the dimension of $S$.

If every ideal $I$ of $S$ is finitely generated, that is, $I = \cup_i(S+s_i)$ for a finite number of elements $s_1, \cdots, s_n$ of $S$, then $S$ is called a Noetherian semigroup.

Many propositions for commutative rings are known to hold for $S$.

For example, if $S$ is a Noetherian semigroup, then every finitely generated extension g-monoid $S[x_1, \cdots, x_n] = S + \sum_i \mathbf{Z}_0 x_i$ is also Noetherin [M3, Proposition 3], and the integral closure of $S$ is a Krull semigroup [M4].

Ideal theory of $S$ is interesting itself and important for semigroup rings.

Let $R$ be a commutative ring, and let $S$ be a g-monoid. There arises the semigroup ring $R[S]$ of $S$ over $R$: $R[S] = R[X;S] = \{\sum_{finite} a_s X^s \mid a_s \in R, s \in S\}$.

If $S$ is the direct sum $\mathbf{Z}_0 \oplus \cdots \oplus \mathbf{Z}_0$ of $n$-copies of $\mathbf{Z}_0$, then $R[S]$ is isomorphic to the polynomial ring $R[X_1, \cdots, X_n]$ of $n$-variables over $R$.

Assume that the semigroup ring $D[S]$ over a domain $D$ is a Krull domain. Then D.F. Anderson [A] and Chouinard [C] showed that $C(D[S]) \cong C(D) \oplus C(S)$, where $C$ denotes ideal class group. Thus they were able to construct Krull domains that have various ideal class groups.

For another example, assume that $D$ is integrally closed and $S$ is integrally closed. Then we have $(I_1 \cap \cdots \cap I_n)^v = I_1^v \cap \cdots \cap I_n^v$ for every finite number of finitely generated ideals $I_1, \cdots, I_n$ of $D[S]$ if and only if $(I_1 \cap \cdots \cap I_n)^v = I_1^v \cap \cdots \cap I_n^v$ for every finite number of finitely generated ideals $I_1, \cdots, I_n$ of $D$ and $(I_1 \cap \cdots \cap I_n)^v = I_1^v \cap \cdots \cap I_n^v$ for every finite number of finitely generated ideals $I_1, \cdots, I_n$ of $S$ ([M1]), where $v$ is the $v$-operation.

## 1

Let $D$ be a Noetherian integral domain with the integral closure $\overline{D}$, and $K$ the quotient field of $D$.

The Krull-Akizuki theorem states that, if $\dim(D) = 1$, then any ring between $D$ and $K$ is Noetherian and its dimension is at most 1.

The Mori-Nagata theorem states that $\overline{D}$ is a Krull ring for any Noetherian domain $D$.

Moreover, Nagata proved that, if $D$ is of dimension 2, then $\overline{D}$ is Noetherian (cf. [Na]).

In [M2] we proved the Krull-Akizuki theorem for semigroups.

In [M4] we proved the Mori-Nagata theorem for semigroups.

Let $T$ be an extension g-monoid of $S$. An element $t$ of $T$ is called integral over $S$ if $nt \in S$ for some positive integer $n$. The set of integral elements of $T$ is called the integral closure of $S$ in $T$. The integral closure $\overline{S}$ in the quotient group $q(S) = \{s - s' \mid s, s' \in S\}$ is called the integral closure of $S$, and is denoted by $\overline{S}$. If $\overline{S} = S$, then $S$ is called integrally closed.

In 1, we proved the following Theorem and answered to the following question in the negative.

**Theorem.** Let $S$ be a 2-dimensional Noetherian semigroup. Then the integral closure $\overline{S}$ of $S$ is a Noetherian semigroup.

Let $P$ be a prime ideal of $S$. Then the maximum number $n$ so that there exists a chain $P_1 \subsetneqq P_2 \subsetneqq \cdots \subsetneqq P_n = P$ of prime ideals of $S$ is called the height of $P$, and is denoted by $ht(P)$.

**Question.** If $P$ is a prime ideal of height $r$ in a Noetherian semigroup $S$, then is $P$ a prime ideal minimal among containing an $r$-generated ideal of $S$?

This is "yes" for rings.

Now, to answer to the Question, let $x_1 + x_2 = x_3 + x_4$ be a unique relation of letters $x_1, x_2, x_3$ and $x_4$. Set $S = \mathbf{Z}_0x_1 + \mathbf{Z}_0x_2 + \mathbf{Z}_0x_3 + \mathbf{Z}_0x_4$. Then $S$ is a g-monoid. $M = (x_1, x_2, x_3, x_4) = \cup_i(S + x_i)$ is a unique maximal ideal of $S$. Then $S$ is a Noetherian semigroup of dimension 3. $M$ is not a prime ideal minimal among containng a 3-generated ideal of $S$.

**2**

Larsen-McCarthy's Multiplicative Theory of Ideals [LM] is one of the basic references of multiplicative ideal theory for commutative rings. In 2, we proved or disproved all the Theorems in [LM] for semigroups. We will state two Theorems.

Let $M$ be a non-empty set. Assume that, for every $s \in S$ and $a \in M$, there is defined $s + a \in M$ such that, for every $s_1, s_2 \in S$ and $a \in M$, we have $(s_1 + s_2) + a = s_1 + (s_2 + a)$ and $0 + a = a$. Then $M$ is called an $S$-module.

**Theorem.** Let $S$ be a Noetherian semigroup, $M$ a finitely generated $S$-module, $L$ and $N$ submodules of $M$, and $I$ an ideal of $S$. Then there exists a positive integer $r$ such that for every $n > r$, we have

$$(nI+L)\cap N=(n-r)I+((rI+L)\cap N).$$

This is a semigroup version of the Artin-Rees Lemma for rings.

Let $M$ be an $S$-module. If $s_1+a=s_2+a$ (for $s_1,s_2\in S$ and $a\in M$) implies $s_1=s_2$, then $M$ is called cancellative.

Theorem implies that if $M$ is a finitely generated cancellative module over a Noetherian semigroup $S$, then $\cap_{n=1}^{\infty}(nI+M)=\emptyset$ for every proper ideal $I$ of $S$.

An element $s$ of a g-monoid $S$ is called unit if $-s\in S$. Let $s$ be a non-unit of $S$. If $s=s_1+s_2$ implies that $s_1$ or $s_2$ is a unit, then $s$ is called irreducible. If every element of $S$ is expressed as a sum of irreducible elements uniquely (up to units and permutation), then $S$ is called factorial (or a UFS).

If there exists a family $\{V_\lambda \mid \lambda\}$ of $\mathbf{Z}$-valued valuation semigroups on $q(S)$ so that $S=\cap_\lambda V_\lambda$ and each element of $S$ is a unit for almost all $\lambda$, then $S$ is called a Krull semigroup.

An $S$-submodule $I$ of $q(S)$ is called a fractional ideal of $S$, if $s+I\subset S$ for some $s\in S$. Let $F(S)$ be the set of fractional ideals of $S$. For every fractional ideal $I$ of $S$, we set $div(I)=\{J\in F(S)\mid J^v=I^v\}$, and set $D(S)=\{div(I)\mid I\in F(S)\}$, and $C(S)=D(R)/\{div(x)\mid x\in q(S)\}$, where $I^v$ is the intersection of principal fractional ideals of $S$ containing $I$. If $I^v=I$, then $I$ is called divisorial.

**Theorem.** If $S$ is a g-monoid, then the following conditions are equivalent:

(1) $S$ is a factorial semigroup.

(2) $S$ is a Krull semigroup and $C(S)=0$.

(3) $S$ is a Krull semigroup and every prime divisorial ideal of $S$ is principal.

**3**

Kaplansky's Commutative Rings [Kap] is one of the basic rerences of commutative ring theory. We know that all the Theorems in Chapters 1 and 2 of [Kap] hold for S [TM].

In 3, we showed that all the Theorems in Chapter 3 of [Kap] hold for g-monoids. We will state some Theorems.

Let $A$ be an $S$-module and $s \in S$. If $s + a_1 = s + a_2$ (for $a_1, a_2 \in A$) implies $a_1 = a_2$, then $s$ is called a non-zerodivisor on $A$. If $s$ is not a non-zerodivisor, then $s$ is called a zerodivisor on $A$. The set of zerodivisors on $A$ is denoted by $Z(A)$. Let $B$ be a submodule of an $S$-module $A$, and $s \in R$. If $s + a \in B$ (for $a \in A$) implies $a \in B$, then $s$ is called a non-zerodivisor on $A$ modulo $B$ (or a non-zerodivisor on $A/B$). If $s$ is not a non-zerodivisor on $A/B$, then $s$ is called a zerodivisor. The set of zerodivisors on $A/B$ is denoted by $Z(A/B)$.

The ordered sequence of elements $x_1, \cdots, x_n$ of $S$ is called a regular sequence on $A$, if $(x_1, \cdots, x_n) + A \subsetneqq A$ and if $x_1 \notin Z(A)$, $x_2 \notin Z(A/((x_1)+A))$, $\cdots$, $x_n \notin Z(A/((x_1, \cdots, x_{n-1}) + A))$.

Let $A$ be an $S$-module. If $Z(A) = \emptyset$, then $A$ is called torsion-free.

Let $A$ be an $S$-module, and $I$ an ideal of $S$. Let $x_1, \cdots, x_n$ be a regular sequence in $I$ on $A$. If $x_1, \cdots, x_n, x$ is not a regular sequence on $A$ for each $x \in I$, then $x_1, \cdots, x_n$ is called a maximal regular sequence in $I$ on $A$.

Let $A$ be an $S$-module, and $I$ an ideal of $S$. Then the maximum of lengths of all regular sequences in $I$ on $A$ is called the grade of $I$ on $A$, and is denoted by $G(I, A)$.

Let $A$ be an $S$-module. If any two maximal regular sequences in $I$ on $A$ have the same length for every ideal $I$ with $I + A \subsetneqq A$, then $A$ is said to satisfy property (*). If $A$ satisfies property (*), we say also that $(S, A)$ satisfies property (*).

**Theorem.** Let $S$ be a Noetherian semigroup, and $A$ a finitely generated torsion-free cancellative $S$-module with property (*). Let $I = (x_1, \cdots, x_n)$ be a proper ideal of $S$. Then $G(I, A) = n$ if and only if $x_1, \cdots, x_n$ is a regular sequence on $A$.

Let $S$ be a Noetherian semigroup with maximal ideal $M$. If $G(M, S) = dim(S)$, then $R$ is called a Macaulay semigroup.

**Theorem.** Let $S$ be a Macaulay semigroup such that $(S, S)$ satisfies property (*). Then we have $G(I,S) = ht(I)$ for every ideal $I$ of $S$.

Let $S$ be a Noetherian semigroup with maximal ideal $M$. The cardinality of a minimal generators of $M$ is called the V-dimension of $S$, and is denoted by $V(S)$.

A Noetherian semigroup $S$ is called a regular semigroup if $V(S) = dim(S)$.

**Theorem.** Let $S$ be a Noetherian semigroup with maximal ideal $M$. Assume that $M$ is generated by a regular sequence $a_1, \cdots, a_k$ on $S$. Then $k = dim(S) = V(S)$, and $S$ is a regular semigroup.

**Theorem.** Any regular semigroup is a Macaulay semigroup.

**Theorem.** The polynomial semigroup $S[X]$ is a Macaulay semigroup if and only if $S$ is a Macaulay semigroup.

4

Let $D$ be an integral domain with quotient field $K$. Let $F(D)$ be the set of non-zero fractional ideals of $D$. A mapping $I \longmapsto I^*$ of $F(D)$ to $F(D)$ is called a star-operation on $D$ if for all $a \in K - \{0\}$ and $I, J \in F(D)$;

(1) $(a)^* = (a)$ and $(aI)^* = aI^*$;
(2) $I \subset I^*$;
(3) If $I \subset J$, then $I^* \subset J^*$; and
(4) $(I^*)^* = I^*$.

Let $\Sigma(D)$ be the set of star-operations on $D$.

Let $F'(D)$ be the set of non-zero $D$-submodules of $K$. A mapping $I \longmapsto I^*$ of $F'(D)$ to $F'(D)$ is called a semistar-operation on $D$ if for all $a \in K - \{0\}$ and $I, J \in F'(D)$;

(1) $(aI)^* = aI^*$;
(2) $I \subset I^*$;
(3) If $I \subset J$, then $I^* \subset J^*$; and
(4) $(I^*)^* = I^*$.

Let $\Sigma'(D)$ be the set of semistar-operations on $D$.

A valuation ring (or a valuation semigroup) $V$ is said to be discrete if its value group is discrete.

In 4, we proved the following Theorems.

**Theorem.** Let $D$ be a domain with dimension $n$. Then $D$ is a discrete valuation ring if and only if $\mid \Sigma'(D) \mid = n+1$.

Let $S$ be a g-monoid with quotient group $G$. A mapping $I \longmapsto I^*$ of $F(S)$ to $F(S)$ is called a star-operation on $S$ if for all $a \in G$, and $I, J \in F(S)$; (1) $(a)^* = (a)$; (2) $(a+I)^* = a+I^*$; (3) $I \subset I^*$; (4) If $I \subset J$, then $I^* \subset J^*$; (5) $(I^*)^* = I^*$.

For example, let $I^v$ be the intersection of principal fractional ideals containing $I$, then $v$ is a star-operation on $S$ which is called the $v$-operation on $S$. Let $\Sigma(S)$ be the set of star-operations on $S$.

Let $F'(S)$ be the set of submodules of $G$. A mapping $I \longmapsto I^*$ of $F'(S)$ to $F'(S)$ is called a semistar-operation on $S$ if, for all $a \in G$ and $I, J \in F'(S)$; (1) $(a+I)^* = a+I^*$; (2) $I \subset I^*$; (3) If $I \subset J$, then $I^* \subset J^*$; (4) $(I^*)^* = I^*$.

Let $\Sigma'(S)$ be the set of semistar-operations on S.

**Theorem.** Let $S$ be a g-monoid with dimension $n$. Then $S$ is a discrete valuation semigroup if and only if $\mid \Sigma'(S) \mid = n+1$.

**Theorem.** Let $V$ be a valuation semigroup of dimension $n$, $v$ its valuation and $\Gamma$ its value group. Let $M = P_n \supsetneqq P_{n-1} \supsetneqq \cdots \supsetneqq P_1$ be the prime ideals of $V$, and let $\{0\} \subsetneqq H_{n-1} \subsetneqq \cdots \subsetneqq H_1 \subsetneqq \Gamma$ be the convex subgroups of $\Gamma$. Let $m$ be a positive integer such that $n+1 \leq m \leq 2n+1$. Then the followings are equivalent:

(1) $\mid \Sigma'(V) \mid = m$.

(2) The maximal ideal of the g-monoid $V_{P_i} = \{s-t \mid s \in V, t \in V-P_i\}$ is principal for exactly $2n+1-m$ of $i$.

(3) The ordered abelian group $\Gamma/H_i$ has a minimal positive element for exactly $2n+1-m$ of $i$.

**Theorem.** Let $V$ be a valuation ring of dimension $n$, $v$ its valuation and $\Gamma$ its value group. Let $M = P_n \supsetneqq P_{n-1} \supsetneqq \cdots \supsetneqq P_1 \supsetneqq (0)$ be the prime ideals of $V$, and let $\{0\} \subsetneqq H_{n-1} \subsetneqq \cdots \subsetneqq H_1 \subsetneqq \Gamma$ be the convex subgroups of $\Gamma$. Let $m$ be a positive integer such that $n+1 \leq m \leq 2n+1$. Then the followings are equivalent:

(1) $| \Sigma'(V) |= m$.

(2) The maximal ideal of $V_{P_i}$ is principal for exactly $2n+1-m$ of $i$.

(3) $\Gamma/H_i$ has a minimal positive element for exactly $2n+1-m$ of $i$.

5

Let $R$ be a commutative ring, and let $K$ be its total quotient ring; $K = \{a/b \mid a \in R, b \text{ is a non-zerodivisor of } R\}$. Let S be a g-monoid, and let G be the quotient group of S.

An element $\alpha \in G$ is called almost integral over $S$ if there exists an element $s$ of $S$ such that $s + n\alpha \in S$ for every positive integer $n$. The set of almost integral elements of $G$ over $S$ is called the complete integral closure (or the CIC) of $S$. If the complete integral closure of $S$ coincides with $S$, then $S$ is called completely integrally closed (or CIC).

$R$ is said to be root closed if whenever $x^n \in R$ for some $x \in K$ and positive integer $n$, then $x \in R$.

The maximal number n so that there exists a set of $n$-elements in $G$ which is independent over $\mathbf{Z}$ is called the torsion-free rank of $G$, and is denoted by t.f.r.$(G)$.

In 5, we proved the following Theorems.

**Theorem.** $R[X;S]$ is integrally closed if and only if $S$ is integrally closed, $R$ is integrally closed, $K[X_1]$ is integrally closed and $q(K[X_1,\cdots, X_{n-1}])[X_n]$ is integrally closed for every $n$ with $n \leq$ t.f.r.$(G)$.

**Theorem.** $R[X;S]$ is CIC if and only if $S$ is CIC, $R$ is CIC and $R[X_1,\cdots,X_n]$ is CIC for every positive integer $n \leq$ t.f.r.$(G)$.

**Theorem.** $R[X;S]$ is root closed if and only if $S$ is integrally closed, $R$ is root closed, $K[X_1]$ is root closed and $q(K[X_1,\cdots,X_{n-1}])[X_n]$ is root closed for every $n$ with $n \leq$ t.f.r.$(G)$.

If, for each element a of $R$, there exists an element $b$ of $R$ such that $a = a^2b$, then $R$ is called a von Neumann regular ring.

**Theorem.** Assume that $K$ is a von Neumann regular ring. Then $R[X;S]$ is integrally closed if and only if $S$ is integrally closed and $R$ is integrally closed.

**Theorem.** Assume that $K$ is a von Neumann regular ring. Then $R[X;S]$ is CIC if and only if $S$ is CIC and $R$ is CIC.

Let $R$ be a Noetherian reduced ring. Then $R[X;S]$ is CIC if and only if $S$ is CIC and $R$ is CIC.

**Theorem.** Assume that $K$ is a von Neumann regular ring. Then $R[X;S]$ is root closed if and only if $S$ is integrally closed and $R$ is root closed.

**6**

We denote the unit group of $S$ by $H$. Let $R$ be a ring. Let $U(R)$ be the unit group of $R$. The group of units $f = \sum a_s X^s$ of $R[X;S]$ with $\sum a_s = 1$ is denoted by $\mathrm{V}(R[X;S])$.

The following is a semigroup version of Karpilovsky's Problem [Kar, chapter 7, problem 9]:

**Problem.** Find necessary and sufficient conditions for $R[X;S]$ under which,

(1) $H$ has a torsion-free complement in $\mathrm{V}(R[X;S])$.
( $V(R[X;S]) = \{X^h \mid h \in H\} \otimes W$, where $W$ is torsion-free.)
(2) $H$ has a free complement in $\mathrm{V}(R[X;S])$.
( $V(R[X;S]) = \{X^h \mid h \in H\} \otimes W$, where $W$ is free.)
(3) $\mathrm{U}(R[X;S])$ is free modulo torsion.
( $U(R[X;S])/\{$torsion elements$\}$ is free.)

In 6, we proved the following,

**Theorem** (An answer to Problem for reduced rings). Let $R$ be reduced. Then,

(1) $H$ has a torsion-free complement in $\mathrm{V}(R[X;S])$.

(2) $H$ has a free complement in $\mathrm{V}(R[X;S])$ if and only if $H$ is free.

(3) $\mathrm{U}(R[X;S])$ is free modulo torsion if and only if $\mathrm{U}(R)$ is free modulo torsion and $H$ is free.

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